# Exercise 1

Use the function $f(z) = (e^{iaz} - e^{ibz})/z^2$ and the indented contour in Fig. 108 (Sec. 89) to derive the integration formula

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\, dx = \frac{\pi}{2}(b - a) \qquad (a \geq 0,\ b \geq 0).$$

Then, with the aid of the trigonometric identity $1 - \cos(2x) = 2\sin^2 x$, point out how it follows that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

---

### Solution

Despite the fact that a singularity exists at $x = 0$, which lies in the interval of integration, the integral converges because the singular point is removable. That is, if the integrand is expanded about $x = 0$,

$$\begin{aligned}\frac{\cos(ax) - \cos(bx)}{x^2} &= \frac{\left[1 - \frac{(ax)^2}{2} + \frac{(ax)^4}{24} - \cdots\right] - \left[1 - \frac{(bx)^2}{2} + \frac{(bx)^4}{24} - \cdots\right]}{x^2} \\ &= \frac{\frac{-a^2+b^2}{2}x^2 + \frac{a^4-b^4}{24}x^4 + \cdots}{x^2} \\ &= \frac{-a^2+b^2}{2} + \frac{a^4-b^4}{24}x^2 + \cdots,\end{aligned}$$

there are no negative powers of $x$. The integrand is an even function, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\, dx = \int_{-\infty}^\infty \frac{\cos(ax) - \cos(bx)}{2x^2}\, dx$$

In order to evaluate the integral, consider the corresponding function in the complex plane, $(e^{iaz} - e^{ibz})/2z^2$, and the contour in Fig. 108.

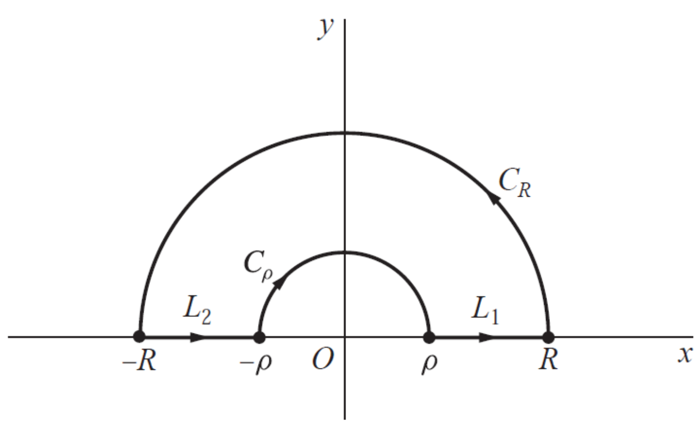


FIGURE 108

Figure 1: The singularity at $z = 0$ is avoided by following the semicircular arc $C_\rho$.

According to Cauchy's residue theorem, the integral of this function around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities $z_n$.

$$\oint_C \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz = 2\pi i \sum_n \operatorname*{Res}_{z=z_n} \frac{e^{iaz} - e^{ibz}}{2z^2}$$

There are no singularities other than $z = 0$, so the right side simplifies to zero.

$$\oint_C \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz = 0$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz + \int_{L_2} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz = 0$$

The parameterizations for the arcs are as follows.

$$\begin{aligned}
L_1 &: \quad z = re^{i0}, && r = \rho \quad \to \quad r = R \\
L_2 &: \quad z = re^{i\pi}, && r = R \quad \to \quad r = \rho \\
C_\rho &: \quad z = \rho e^{i\theta}, && \theta = \pi \quad \to \quad \theta = 0 \\
C_R &: \quad z = Re^{i\theta}, && \theta = 0 \quad \to \quad \theta = \pi
\end{aligned}$$

As a result,

$$\begin{aligned}
0 &= \int_\rho^R \frac{e^{iare^{i0}} - e^{ibre^{i0}}}{2(re^{i0})^2}(dr\,e^{i0}) + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz + \int_R^\rho \frac{e^{iare^{i\pi}} - e^{ibre^{i\pi}}}{2(re^{i\pi})^2}(dr\,e^{i\pi}) + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz \\
&= \int_\rho^R \frac{e^{iar} - e^{ibr}}{2r^2}\,dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz + \int_R^\rho \frac{e^{-iar} - e^{-ibr}}{2(-r)^2}(-dr) + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz \\
&= \int_\rho^R \frac{e^{iar} - e^{ibr}}{2r^2}\,dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz + \int_\rho^R \frac{e^{-iar} - e^{-ibr}}{2r^2}\,dr + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz \\
&= \int_\rho^R \frac{e^{iar} + e^{-iar} - e^{ibr} - e^{-ibr}}{2r^2}\,dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz \\
&= \int_\rho^R \frac{(2\cos ar) - (2\cos br)}{2r^2}\,dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz \\
&= \int_\rho^R \frac{\cos ar - \cos br}{r^2}\,dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\,dz.
\end{aligned}$$

Take the limit now as $\rho \to 0$ and $R \to \infty$. The integral over $C_\rho$ tends to $(-\pi/2)(b-a)$, and the integral over $C_R$ tends to zero. Proof for these statements will be given at the end.

$$\int_0^\infty \frac{\cos ar - \cos br}{r^2}\,dr + \left(-\frac{\pi}{2}\right)(b-a) = 0$$

Change the dummy integration variable to $x$ and solve for the integral. Therefore,

$$\boxed{\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a).}$$

Note that this formula holds for $a \geq 0$ and $b \geq 0$, so to obtain the second result, set $a = 0$ and $b = 2$.

$$\int_0^\infty \frac{\cos 0 - \cos 2x}{x^2}\,dx = \frac{\pi}{2}(2)$$

$$\int_0^\infty \frac{1 - \cos 2x}{x^2}\,dx = \pi$$

Apply the provided trigonometric identity $1 - \cos 2x = 2\sin^2 x$.

$$\int_0^\infty \frac{2\sin^2 x}{x^2}\,dx = \pi$$

Divide both sides by 2. Therefore,

$$\boxed{\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.}$$

**The Integral Over $C_\rho$**

Our aim here is to show that the integral over $C_\rho$ tends to $(-\pi/2)(b-a)$ in the limit as $\rho \to 0$. The parameterization of the small semicircular arc in Fig. 108 is $z = \rho e^{i\theta}$, where $\theta$ goes from $\pi$ to 0.

$$\begin{aligned}\int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz &= \int_\pi^0 \frac{e^{ia\rho e^{i\theta}} - e^{ib\rho e^{i\theta}}}{2(\rho e^{i\theta})^2}(\rho i e^{i\theta}\, d\theta) \\ &= \int_\pi^0 \frac{e^{ia\rho e^{i\theta}} - e^{ib\rho e^{i\theta}}}{2\rho e^{i\theta}}(i\, d\theta)\end{aligned}$$

In the limit as $\rho \to 0$, we have

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz = \lim_{\rho\to 0}\int_\pi^0 \frac{e^{ia\rho e^{i\theta}} - e^{ib\rho e^{i\theta}}}{2\rho e^{i\theta}}(i\, d\theta).$$

Since the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz = \int_\pi^0 \lim_{\rho\to 0} \frac{e^{ia\rho e^{i\theta}} - e^{ib\rho e^{i\theta}}}{2\rho e^{i\theta}}(i\, d\theta)$$

Plugging in $\rho = 0$ yields the indeterminate form 0/0, so l'Hôpital's rule will be applied to calculate the limit.

$$\begin{aligned}\lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz &\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \int_\pi^0 \lim_{\rho\to 0} \frac{iae^{i\theta}e^{ia\rho e^{i\theta}} - ibe^{i\theta}e^{ib\rho e^{i\theta}}}{2e^{i\theta}}(i\, d\theta) \\ &= \int_\pi^0 \lim_{\rho\to 0} \frac{iae^{ia\rho e^{i\theta}} - ibe^{ib\rho e^{i\theta}}}{2}(i\, d\theta) \\ &= \int_\pi^0 \frac{ia - ib}{2}(i\, d\theta) = \int_\pi^0 \frac{b-a}{2}\, d\theta\end{aligned}$$

Therefore,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz = -\frac{\pi}{2}(b-a).$$

**The Integral Over $C_R$**

Our aim here is to show that the integral over $C_R$ tends to zero in the limit as $R \to \infty$. The parameterization of the large semicircular arc in Fig. 108 is $z = Re^{i\theta}$, where $\theta$ goes from 0 to $\pi$.

$$\begin{aligned}\int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz &= \int_0^\pi \frac{e^{iaRe^{i\theta}} - e^{ibRe^{i\theta}}}{2(Re^{i\theta})^2}(Rie^{i\theta}\, d\theta) \\ &= \int_0^\pi \frac{e^{iaRe^{i\theta}} - e^{ibRe^{i\theta}}}{2Re^{i\theta}}(i\, d\theta) \\ &= \int_0^\pi \frac{e^{iaR(\cos\theta + i\sin\theta)} - e^{ibR(\cos\theta + i\sin\theta)}}{2Re^{i\theta}}(i\, d\theta) \\ &= \int_0^\pi \frac{e^{iaR\cos\theta}e^{-aR\sin\theta} - e^{ibR\cos\theta}e^{-bR\sin\theta}}{2Re^{i\theta}}(i\, d\theta)\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}
\left|\int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz\right| &= \left|\int_0^\pi \frac{e^{iaR\cos\theta}e^{-aR\sin\theta} - e^{ibR\cos\theta}e^{-bR\sin\theta}}{2Re^{i\theta}}(i\, d\theta)\right| \\
&\leq \int_0^\pi \left|\frac{e^{iaR\cos\theta}e^{-aR\sin\theta} - e^{ibR\cos\theta}e^{-bR\sin\theta}}{2Re^{i\theta}}(i)\right| d\theta \\
&= \int_0^\pi \frac{\left|e^{iaR\cos\theta}e^{-aR\sin\theta} - e^{ibR\cos\theta}e^{-bR\sin\theta}\right|}{|2Re^{i\theta}|}\, |i|\, d\theta \\
&= \int_0^\pi \frac{\left|e^{iaR\cos\theta}e^{-aR\sin\theta} - e^{ibR\cos\theta}e^{-bR\sin\theta}\right|}{2R}\, d\theta \\
&\leq \int_0^\pi \frac{\left|e^{iaR\cos\theta}e^{-aR\sin\theta}\right| + \left|e^{ibR\cos\theta}e^{-bR\sin\theta}\right|}{2R}\, d\theta \\
&= \int_0^\pi \frac{e^{-aR\sin\theta} + e^{-bR\sin\theta}}{2R}\, d\theta
\end{aligned}$$

So we have

$$\left|\int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz\right| \leq \int_0^\pi \frac{e^{-aR\sin\theta} + e^{-bR\sin\theta}}{2R}\, d\theta.$$

Take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz\right| \leq \lim_{R\to\infty}\int_0^\pi \frac{e^{-aR\sin\theta} + e^{-bR\sin\theta}}{2R}\, d\theta$$

Because the limits are constant, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz\right| \leq \int_0^\pi \lim_{R\to\infty}\frac{e^{-aR\sin\theta} + e^{-bR\sin\theta}}{2R}\, d\theta$$

Since $\theta$ lies between 0 and $\pi$, the sine of $\theta$ is positive. In addition, $R$, $a$, and $b$ are nonnegative, so each of the exponential functions tends to zero (or to one if $a = 0$ or $b = 0$). In any case, the integral tends to zero because of the $R$ in the denominator.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz\right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2}\, dz = 0.$$